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CALCULATION OF THE RISE VELOCITY OF LARGE GAS BUBBLES

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The problem of determining the rate of rise of large gas bubbles in an infinite volume and a vertical tube is considered. In the first case the shape of the bubble is constructed by "gluing together" the front surface immersed in a flow of ideal liquid and the rear surface being formed under the action of gravitational and capillary forces, and in the second case the solution of the Laplace equation is constructed for the velocity potential in the problem of ideal liquid flow in a tube around a semiinfinite cylindrical body with a hemispherical frontal part.

The problem of determining the rise velocity of gas bubbles in a liquid that fills an infinite volume and a vertical tube is a completely independent problem of the mechanics of two-phase flows [1–3].

Results of numerous experiments [1–3] show that the rise velocity of sufficiently large bubbles is independent of the liquid viscosity. Consequently, the problem of the rise of a bubble in a liquid can be reduced to the problem of the flow of an ideal liquid past a free surface in a gravity field with the velocity at infinity U_∞ (equal to the bubble rise velocity in a quiescent liquid).

Theoretical solutions of the problem of the rise of a bubble in an infinite volume cover only the region of small bubbles [3]. Well-known theoretical investigations of the rise of a bubble in a vertical tube [4, 5] use sums of the first few terms of infinite exponential series along the axial coordinate of the flow for the infinite interval $(-\infty, +\infty)$. These sums diverge on one of the semi-infinite intervals.

In the present study, which is a continuation of [6–8], we obtained approximate analytical solutions that determine the shape and the rise velocity of large gas bubbles.

1. Rise of a Gas Bubble in an Infinite Volume. As is known [1–3], bubbles whose equivalent radius is much larger than the capillary constant ("very large" bubbles) have a shape close to a spherical segment (Fig. 1). Their rise velocity U_∞ is related to the equivalent radius R_{eq} and the gravitational acceleration g by the relation [1–3]:

$$U_\infty = k_1 \sqrt{gR_{\text{eq}}}\,, \tag{1}$$

where $k_1 \approx 0.95–1.05$.

The combined use of the Bernoulli equation for the streamline passing through the front stagnation point, the condition $P = \text{const}$ on the free surface, and the law of ideal liquid flow around an arbitrary sphere (the upper segment of which is occupied by the bubble) gives the following relationship between the radius of the segment R and the liquid velocity at infinity:

$$R = \frac{9}{4}\frac{U_\infty^2}{g}. \tag{2}$$

Formula (2) was obtained already in [5]; when the value of the height of the segment h is unknown (see Fig. 1), it does not permit one to relate the rise velocity of the bubble to its equivalent radius R_{eq}. To close the description

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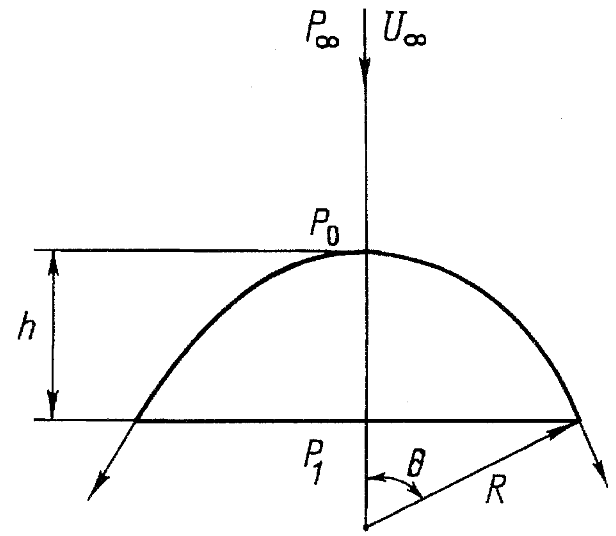

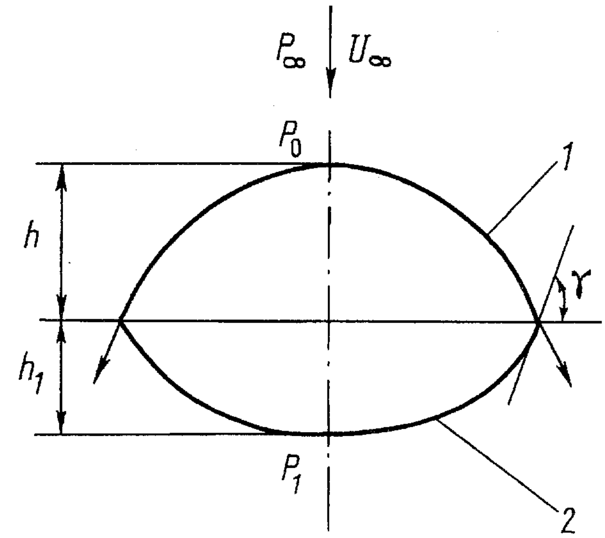

Fig. 1. Scheme of flow round a large gas bubble.

Fig. 2. Scheme of flow round a moderately large gas bubble: 1) the front surface of the bubble immersed in the flow; 2) hydrostatic rear surface of the bubble.

of the problem, the authors of [6] used an empirical relation from [9] for separation flow past bodies of circular midsection:

$$P_0 - P_1 = \frac{1+\theta}{2}\rho U_\infty^2 , \tag{3}$$

where P_0, P_1 are the pressure at the front stagnation point and in the "wake region" behind the body; $\theta = 0.4$ is the coefficient of "bottom rarefaction" [9].

Superposition of the gravity field and the condition $P = \text{const}$ on the pattern of separation flow for the entire surface of the bubble leads to the following relationship between the height of the segment h and the liquid velocity at infinity U_∞:

$$h = \frac{1+\theta}{2}\frac{U_\infty^2}{g} . \tag{4}$$

Formulas (2), (4) and elementary geometric relations for a spherical segment yield the final formula

$$U_\infty = 1.05\sqrt{gR_{eq}} , \tag{5}$$

which virtually coincides with Eq. (1).

Photographs in [3] show that as the size of the bubble decreases its "rear" surface becomes more and more convex, while the shape of a "moderately large" bubble acquires the shape of a prolate (in the direction of motion) ellipsoid. Within the framework of the scheme of separation flow the rear surface of the bubble should be determined as a part of the equilibrium hydrostatic surface formed under the simultaneous effect of gravity and capillary forces. Thus, the shape of a "moderately large" bubble is constructed by "gluing" two parts: the front part immersed in the flow and the hydrostatic rear part (Fig. 2). It should be noted that the real surface of a bubble cannot have sharp protrusions. The use of the scheme of Fig. 2 is justifiable for the following reasons: 1) the "gluing" is made along a circle lying on the surface of separation streamlines, i.e., immediatelt inside the discontinuity, which is considered to be infinitely thin in the well-known scheme of separation flow [9]; 2) when an "acute" surface is replaced by a "rounded" one with a small radius the geometric relations used in the calculations are virtually unchanged.

In [10] a theoretical determination is given for a family of surfaces of a bubble bounded by a liquid from below and by a solid wall from above and containing the contact angle γ as a parameter. The assignment of a similar quantity for the case of "gluing" the front and rear surfaces of a "moderately large" bubble and account for Laplace jumps in pressure on the upper and lower surfaces of the bubble lead to the following relation [7]:

$$\frac{1+\theta}{2}\rho g R = \rho g (h + h_1) + 2\sigma \left(\frac{1}{R_1} - \frac{1}{R} \right), \tag{6}$$

where h_1 and R_1 are the height of the rear portion of the bubble and the radius of curvature at its lower point; σ is the coefficient of surface tension. In [7] relation (6) was used at the value of the "gluing angle" $\gamma \approx \pi/4$. It seems physically more justifiable to use the relation

$$h + h_1 = k_1 R_1, \tag{7}$$

which assumes that the vertical and horizontal dimensions of the bubble (similarly to the axes of an ellipsoid of revolution) preserve a fixed relation for the entire range of the dimensions of the bubble investigated.

Using the results of a theoretical solution [10] for the equilibrium hydrostatic surface and also neglecting in Eq. (6) the quantity $1/R \ll 1/R_1$, we obtain the final relation for the rise velocity of a bubble in a pool of liquid, which is valid for the entire investigated range of the dimensions of the bubbles (from "moderately large" to "very large"):

$$U_\infty^2 = k_1 g R_{eq} + k_2 \frac{\sigma}{\rho R_{eq}}. \tag{8}$$

The use of empirical relation (7) alone permits one to determine both constants in Eq. (8): $k_1 \approx 1.0$; $k_2 \approx 1.3$. Thus, relation (8) virtually coincides with the well-known empirical formula from [11].

2. Rise of a Bubble in a Vertical Tube. Experiments show [4, 5] that a large bubble rising in a vertical tube has an approximately hemispherical front portion passing into a cylindrical "periphery" separated from the wall by a thin liquid layer. The rise velocity of the bubble is independent of its length (which is much larger than the radius of the tube) and is determined only by the tube radius R_0:

$$U_\infty = k_3 \sqrt{g R_0}, \tag{9}$$

where $k_3 \approx 0.48-0.50$. The flow around a semiinfinite body in a tube that simulates a bubble is described by the Laplace equation for the velocity potential φ

$$\frac{\partial^2 \varphi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \varphi}{\partial r} \right) = 0, \tag{10}$$

where z, r are the dimensionless axial and radial coordinates (the tube radius R_0 is the length scale). Solution (10) is sought in the form of a superposition of two flows (Fig. 3):

a) the flow from a point source in an impenetrable tube located at the coordinate origin ($z = r = 0$) that is a homogeneous flow with the velocity U_0 at $z \to \pm\infty$;

b) a "superimposed" homogeneous flow with the velocity $U_1 \geq U_0$.

Let us write out the dimensionless expressions for the velocity potential φ and the axial u and radial v velocities of the flow (the asymptotic velocity of the source U_0 is taken as the velocity scale):

$$\varphi = -\frac{1}{2}\frac{1}{(z^2 + r^2)^{1/2}} - \frac{1}{\pi}\int_0^\infty \frac{K_1(\varepsilon) I_0(\varepsilon r)\cos(\varepsilon z)}{I_1(\varepsilon)} d\varepsilon - fz; \tag{11}$$

$$u = \frac{1}{2}\frac{z}{(z^2 + r^2)^{3/2}} + \frac{1}{\pi}\int_0^\infty \frac{\varepsilon K_1(\varepsilon) I_0(\varepsilon r)\sin(\varepsilon z)}{I_1(\varepsilon)} d\varepsilon - f; \tag{12}$$

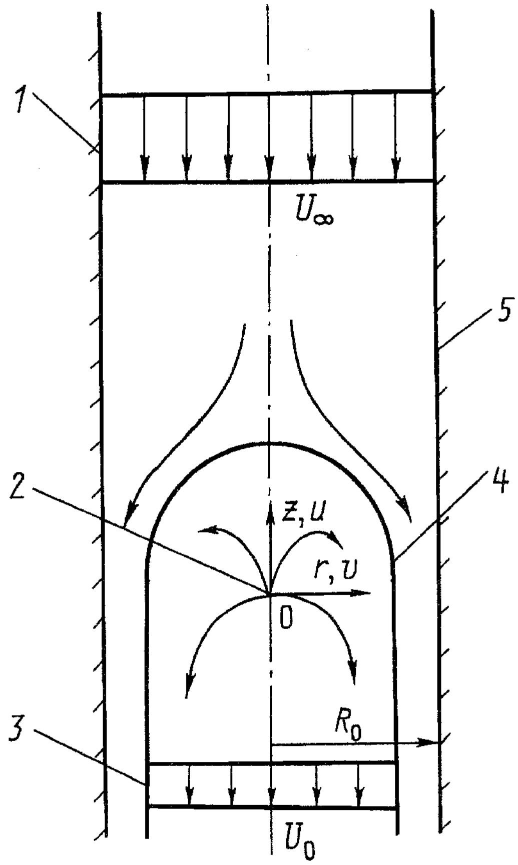

Fig. 3. Scheme of the flow past a large gas bubble in a cylindrical tube: 1) overall velocity of the flow from the mass source U_0 and the imposed uniform flow U_1 for $z \to \infty$ (the rise velocity of the bubble in the tube); 2) point mass source at the coordinate origin; 3) asymptotic velocity of the flow from the mass source for $z \to -\infty$; 4) surface of the bubble; 5) surface of the tube.

$$v = \frac{1}{2}\frac{r}{(z^2 + r^2)^{3/2}} - \frac{1}{\pi}\int_0^\infty \frac{\varepsilon K_1(\varepsilon) I_1(\varepsilon r)\cos(\varepsilon z)}{I_1(\varepsilon)}\, d\varepsilon\,. \tag{13}$$

Here $f = U_1/U_0$ is the source intensity parameter, connected with the distance z_0 from the coordinate origin to the front stagnation point of the body in flow by the relation

$$f = \frac{1}{2z_0^2} + \frac{1}{\pi}\int_0^\infty \frac{\varepsilon K_1(\varepsilon)\sin(\varepsilon z_0)}{I_1(\varepsilon)}\, d\varepsilon \tag{14}$$

($I_0(\varepsilon)$, $I_1(\varepsilon)$, $K_1(\varepsilon)$ are modified Bessel functions of the first and second kind [2]). The flow described by relations (11)-(14) satisfies the condition of zero leakage on the tube wall (at $r = 1$) and also on the surface of the body immersed in flow (see Fig. 3). Representing the flow velocity components by linear terms of an expansion in a Taylor series in the vicinity of the front stagnation point with account for the condition $P = \text{const}$ on the free surface, we obtain

$$\frac{U_\infty^2}{gR_0} = -\frac{(f-1)^2 f''}{(f')^3}. \tag{15}$$

Here $f' = df/dz_0$, $f'' = d^2f/dz_0^2$, $U_\infty = U_1 - U_0$ is the overall velocity of the flow for $z \to \infty$ (equal to the bubble rise velocity in the tube).

To close the description of the problem the simplest version of the method of collocations [13] was used in [8]: it was necessary to fulfill the condition $P = \text{const}$ also at the point on the body surface at $z = 0$ in addition to the front stagnation point.

To find a more rigorous closure of the problem, we will apply the above-described approximate method to the problem of the rise of a "plane bubble" in a slit described by the two-dimensional Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \tag{16}$$

where x is the transverse coordinate reckoned from the symmetry plane (here the length scale H is equal to half the distance between the two planes that form the slit). The problem of the rise of a "plane bubble" has an abstract nature and is not realized experimentally; it has the advantage of having an exact solution [9]:

$$U_\infty = 0.345\sqrt{gH}. \tag{17}$$

Let us write out an approximate solution for the plane problem similar to solution (11)-(15) for the axisymmetric problem:

$$\varphi = \frac{1}{\pi}\ln(z^2 + x^2) - \frac{2}{\pi}\int_0^\infty \frac{\exp(-\varepsilon)\,\text{ch}(\varepsilon x)\cos(\varepsilon z)}{\varepsilon\,\text{sh}\,\varepsilon}\,d\varepsilon - fz; \tag{11'}$$

$$u = \frac{2}{\pi}\frac{z}{z^2 + x^2} + \frac{2}{\pi}\int_0^\infty \frac{\exp(-\varepsilon)\,\text{ch}(\varepsilon x)\sin(\varepsilon z)}{\text{sh}\,\varepsilon}\,d\varepsilon - f; \tag{12'}$$

$$v = \frac{2}{\pi}\frac{x}{z^2 + x^2} - \frac{2}{\pi}\int_0^\infty \frac{\exp(-\varepsilon)\,\text{sh}(\varepsilon x)\cos(\varepsilon z)}{\text{sh}\,\varepsilon}\,d\varepsilon; \tag{13'}$$

$$f = \text{cth}\left(\frac{\pi}{2}z_0\right); \tag{14'}$$

$$\frac{U_\infty^2}{gH} = \frac{(1 - \exp(-\pi z_0))^2}{3\pi}. \tag{15'}$$

In the limit $z_0 \to \infty$ Eq. (15′) yields a relation for the rise velocity of a "plane bubble" $U_\infty/\sqrt{gH} = 0.326$, which agrees well with the exact solution of [9].

Performing a similar limiting transition for $z_0 \to \infty$ for the problem of the rise of a bubble in a tube (the formal mathematical procedure of which is described in [14]), we arrive at the final relation for the rise velocity:

$$\frac{U_\infty}{\sqrt{gR_0}} \approx 0.511, \tag{18}$$

which agrees well with empirical formula (9).

Expressions (8) and (18), which determine the rise velocity of large gas bubbles in a liquid filling an infinite volume and a round tube, can be used in various applications of the mechanics of two-phase flows.

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